Best approximation and optimal location in polyhedral Banach spaces

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Convexity in Banach Spaces
A homage to Piero Papini
Castro Urdiales, Spain

Works of P.L.P. on polyhedral Banach spaces:


Work of P.L.P. on optimal location:

Notation

$X \ldots$ real Banach space, $Y \subset X \ldots$ closed subspace

For $x \in X$ consider the nearest point set

$$P_Y(x) = \{ y \in Y : \|x - y\| = d(x, Y) \}.$$
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$P_Y : X \to 2^Y$ (metric projection)

$\text{dom}(P_Y) = \{x \in X : P_Y(x) \neq \emptyset\}$ (effective domain of $P_Y$)
Hausdorff upper and lower semicontinuity

$T \ldots$ topological space, $Z \ldots$ normed space, $t_0 \in T$, $F: T \rightarrow 2^Z$
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$F$ is Hausdorff u.s.c. (H-u.s.c.) at $t_0$ if

$$\forall \varepsilon > 0 \exists V \in \mathcal{U}(t_0) : F(t) \subset F(t_0) + \varepsilon B_0^Z$$

whenever $t \in V$.

Observation

u.s.c. $\Rightarrow$ H-u.s.c.; H-l.s.c. $\Rightarrow$ l.s.c.;

$[\text{H-u.s.c.} + \text{H-l.s.c.}] \iff$ contin. in the Hausdorff (pseudo)metric.
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u.s.c. \( \Rightarrow \) H-u.s.c.; H-l.s.c. \( \Rightarrow \) l.s.c.;

\[ \text{[H-u.s.c. + H-l.s.c.] } \iff \text{contin. in the Hausdorff (pseudo)metric.} \]
\[ F \text{ is l.s.c. at } t_0 \; \iff \quad \forall \varepsilon > 0 \; \forall z_0 \in F(t_0) \; \exists V = V(\varepsilon, z_0) \in \mathcal{U}(t_0): \quad F(t) \cap B^0(z_0, \varepsilon) \neq \emptyset \quad (t \in V). \]

\[ F \text{ is H-l.s.c. at } t_0 \; \iff \quad \forall \varepsilon > 0 \; \exists V = V(\varepsilon) \in \mathcal{U}(t_0): \quad F(t) \cap B^0(z_0, \varepsilon) \neq \emptyset \quad (t \in V, \; z_0 \in F(t_0)). \]
Polyhedrality

A Banach space $X$ is *polyhedral* if the unit ball of each of its finite-dimensional (equivalently, two-dimensional) subspaces is a polytope. We shall write: $X$ has $(P)$. 

Obvious: $(P)$ is hereditary to subspaces.

A canonical example: (any subspace of) any $c_0(\Gamma)$ space is polyhedral. Easy exercise: the sequence space $c_0$ is not polyhedral. (But it is isomorphically polyhedral.)
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Easy exercise: the sequence space $c$ is not polyhedral. (But it is isomorphically polyhedral.)
A boundary for $X$ is a set $B \subset S_{X^*}$ such that $\|x\| = \max_{f \in B} f(x)$ for each $x \in X$. That is,

$$\bigcup_{f \in B} [f^{-1}(1) \cap B_x] = S_X.$$
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**Theorem (Fonf)**

If $X$ has (P), then:

(a) $S_X$ is covered by “true faces” of $B_X$;
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(c) $X$ is an Asplund space;
(d) $B_{X^*} = \overline{\text{conv} \| \cdot \| \mathcal{B}}$ for every boundary $\mathcal{B} \subset S_{X^*}$. 

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Best approximation and opt. location in polyhedral B. spaces
If \( X \) has \((P)\), the set \( B_0 := w^*\text{-exp} \ B_X^* = w^*\text{-str} \exp B_X^* \) is the minimal boundary.

**Definition**

\( X \) has \((P)\) if there exists a boundary \( B \subset S_X^* \) such that \( f(x) < 1 \) whenever \( x \in S_X^* \) and \( f \) is a \( w^* \)-cluster point of \( B \).

\( X \) has \((P) \Delta\) if \( X \) has \((P)\) and there exists a boundary \( B \subset S_X^* \) such that \( \text{card} \{ b \in B : b(x) = 1 \} \) is finite for each \( x \in S_X^* \).

In both properties, we can equivalently consider the particular boundary \( B = \text{ext} B_X^* \) or, in the class of polyhedral spaces, \( B = B_0 \).

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If $X$ has $(P)$, the set $\mathcal{B}_0 := w^*-\exp B_{X^*} = w^*-\text{str exp } B_{X^*}$ is the minimal boundary.

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In both properties, we can equivalently consider the particular boundary \( \mathcal{B} = \text{ext } B_{X^*} \) or, in the class of polyhedral spaces, \( \mathcal{B} = \mathcal{B}_0 \).
Basic implications:

\[ [\text{subspace of } c_0(\Gamma)] \Rightarrow (\star) \Rightarrow (P\Delta) \Rightarrow (P) \]
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For \( X \) separable, the properties \( (\ast) \), \( (P\Delta) \) and \( (P) \) are isomorphically equivalent (Fonf).
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For \(X\) separable, the properties (\(\ast\)), (\(P\Delta\)) and (\(P\)) are isomorphically equivalent (Fonf).

The properties (\(\ast\)) and (\(P\Delta\)) are hereditary to closed subspaces.

\(X\) has (\(P\Delta\)) \(\Rightarrow\) \(X\) is quasi-polyhedral, i.e.,

\[
\forall x \in S_X \exists V \in \mathcal{U}(x) \ \forall y \in V \cap S_X : [x, y] \subset S_X.
\]
Theorem ([FLV])

Let $Y$ be a closed subspace of $X$.

(a) If $X$ has ($P\Delta$), then $P_Y$ is H-l.s.c. on its effective domain.
(In particular, $P_Y|_{\text{dom}(P_Y)}$ admits a continuous selection.)

(b) If $X$ has ($\ast$), then $P_Y$ is Hausdorff continuous on $\text{dom}(P_Y)$. 
Optimal location (generalized centers of finite sets)

Given \( a = (a_1, \ldots, a_n) \in X^n \) and \( f : \mathbb{R}^n_+ \to \mathbb{R} \), we want to minimize \( \phi(x) = f(\|x - a_1\|, \ldots, \|x - a_n\|) \) \((x \in X)\).

\[ E(f) = \{ x \in X : \phi(x) = \text{inf}_{X} \phi \} \]

(set of \( f \)-centers of \( a \)).

Definition \( X \) has (GC) if \( E(f(a)) \neq \emptyset \) whenever \( n \in \mathbb{N}, a \in X^n \) and \( f : \mathbb{R}^n_+ \to \mathbb{R} \) is continuous, nondecreasing, coercive.

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**Definition**

\(X\) has \((GC)\) if

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whenever \(n \in \mathbb{N}, a \in X^n\) and \(f : \mathbb{R}_+^n \to \mathbb{R}\) is continuous, nondecreasing, coercive.
Proposition (L.V., 1997)

In the definition of (GC) we can equivalently consider only the functions \( f \) of type

\[
f(t_1, \ldots, t_n) = \max_{1 \leq i \leq n} \alpha_i t_i
\]

with \( \alpha_i > 0 \) for each \( i \).
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1. In the definition of $(GC)$ we can equivalently consider only the functions $f$ of type

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2. The following spaces $X$ have $(GC)$:

- $X$ norm-one complemented in $X^{**}$ (e.g., dual, $L_1(\mu)$);
- $X = c_0(\Gamma)$;
- $X = C_b(T, Z)$ where $T$ is a topological space, $\dim Z < \infty$ and $Z \in \{"strictly convex", "polyhedral", "2-dimensional"\}.$
Simple but key observation:

If $\pi : \mathbb{R}^n \to \mathbb{R}$ is a lattice norm, then

$$\varphi(x) := \pi(\|x - a_1\|, \ldots, \|x - a_n\|) = \|(x, \ldots, x) - (a_1, \ldots, a_n)\|.$$ 

Thus

$$E_{\pi}(a) = P_D(a)$$

where $D$ is the “diagonal” \{(x, \ldots, x) : x \in X\} \subset (X^n, \| \cdot \|).$
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Thus

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where \( D \) is the “diagonal” \( \{(x, \ldots, x) : x \in X\} \subset (X^n, \| \cdot \|) \).

Key question:

when \( (X^n, \| \cdot \|) \) satisfies \( (GC)/(P\Delta)/(\ast) \)?
Theorem

Let \((X, \| \cdot \|)\) be a Banach space, \(\pi\) a polyhedral lattice norm on \(\mathbb{R}^n\). Consider \(X^n\) equipped with the norm
\[
\|(x_1, \ldots, x_n)\| = \pi(\|x_1\|, \ldots, \|x_n\|).
\]

1. \(X^n\) has (GC) \(\iff\) \(X\) has (GC).

A norm \(\pi\) on \(\mathbb{R}^n\) is "handy" if:
\[
\forall i \in \{1, \ldots, n\} \forall t \in \mathbb{R}^n \text{ with } t_i = 0 : 
\pi(t + \tau e_i) = \pi(t)
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whenever \(\tau \in \mathbb{R}\) is sufficiently small.
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Theorem

Let \((X, \| \cdot \|)\) be a Banach space, \(\pi\) a polyhedral lattice norm on \(\mathbb{R}^n\). Consider \(X^n\) equipped with the norm \[\|(x_1, \ldots, x_n)\| = \pi(\|x_1\|, \ldots, \|x_n\|).\]

1. \(X^n\) has (GC) \(\iff\) \(X\) has (GC).
2. \(X^n\) has (P) \(\iff\) \(X\) has (P).
3. \(X^n\) has (PΔ) \(\iff\) \(X\) has (PΔ) and: either \(\dim X < \infty\) or \(\pi\) is “handy”.

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4. \(X^n\) has (*) \iff \(X\) has (*) and: either \(\dim X < \infty\) or \(\pi\) is "handy".
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whenever \(\tau \in \mathbb{R}\) is sufficiently small.
Example.
$c_0 \oplus_1 \mathbb{R}$ has neither $(P \Delta)$ nor $(*)$. 
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\textbf{Proposition}

\textit{The classes }$(GC)$, $(P)$, $(P\Delta)$, $(\ast)$\textit{ are closed under making arbitrary $c_0$-sums.}
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Proposition

The classes (GC), (P), (P\Delta), (*) are closed under making arbitrary c_0-sums.

Theorem (easy)

Let X have (GC). Let \( \pi \) be a polyhedral lattice norm on \( \mathbb{R}^n \).
Suppose that either \( \dim X < \infty \) or \( \pi \) is “handy”.

1. If X has (P\Delta), then \( E_\pi(\cdot) \) is H-l.s.c. on \( X^n \).
2. If X has (*), then \( E_\pi(\cdot) \) is H-continuous on \( X^n \).
Theorem (generalization to relative centers)

Let $Y$ be a closed subspace of $X$. For $a \in X^n$ and $f: \mathbb{R}^n_+ \to \mathbb{R}$, consider the set of relative $Y$-centers ("$(f, Y)$-centers") of $a$

$$E_{f,Y}(a) = \{y \in Y : \varphi(y) = \inf \varphi(Y)\}$$

(where $\varphi(x) = f(\|x - a_1\|, \ldots, \|x - a_n\|)$).

Suppose that $X$ has $(\ast)$ and every $a \in X^n$ admits weighted Chebyshev $Y$-centers for all weights $\alpha \in (0, \infty)^n$. Then:

1. $E_{f,Y}(a) \neq \emptyset$ whenever $a \in X^n$ and $f$ is continuous, nondecreasing and coercive;

2. for each $\pi$ polyhedral lattice norm on $\mathbb{R}^n$, the $(\pi, Y)$-center map $E_{\pi,Y}(\cdot)$ is $H$-continuous on $X^n$. 

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Theorem (easy, too)

If $X$ has $(GC)$ and $(P\Delta)$, then $C_b(T,X)$ has $(GC)$. 

Corollary

Assume: $X$ has $(GC)$ and $(\ast)$; $\pi$ is a polyhedral lattice norm on $\mathbb{R}^n$; $K$ is a Hausdorff compact. Let either $\dim X < \infty$ or $\pi$ is "handy". Then the $\pi$-center map $E_{\pi}: C(K,X)^n \to C(K,X)$ is (nonempty-valued and) Hausdorff continuous.

Assume: $X$ has $(GC)$ and $(\ast)$; $K$ is a Hausdorff compact. Then, for each $m \in \mathbb{N}$, the Chebyshev center map $A \mapsto Z(A)$ of $C(K,X)$ is $H$-continuous on the family $P_m := \{ A \subset C(K,X) : \text{card} A \leq m \}$. 

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If $X$ has (GC) and ($P\Delta$), then $C_b(T, X)$ has (GC).

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Assume: $X$ has (GC) and ($\ast$); $\pi$ is a polyhedral lattice norm on $\mathbb{R}^n$; $K$ is a Hausdorff compact. Let either $\dim X < \infty$ or $\pi$ is “handy”. Then the $\pi$-center map

$$E_\pi : C(K, X)^n \to 2^{C(K,X)}$$

is (nonempty-valued and) Hausdorff continuous.
**Theorem (easy, too)**

If $X$ has (GC) and $(P\Delta)$, then $C_b(T, X)$ has (GC).

**Theorem**

Assume: $X$ has (GC) and $(*);$ $\pi$ is a polyhedral lattice norm on $\mathbb{R}^n;$ $K$ is a Hausdorff compact. Let either $\dim X < \infty$ or $\pi$ is “handy”. Then the $\pi$-center map

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**Corollary**

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Thank you for your attention!