On the nonexistence of almost Moore digraphs of diameter four *

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Abstract. Almost Moore digraphs appear in the context of the degree/diameter problem as a class of extremal directed graphs, in the sense that their order is one less than the unattainable Moore bound $M(d,k) = 1 + d + \cdots + d^k$, where $d > 1$ and $k > 1$ denote the maximum out-degree and diameter, respectively. So far, the problem of their existence has only been solved when $d = 2, 3$ or $k = 2, 3$. In this paper we deal with the case of almost Moore digraphs of diameter $k = 4$. Their construction turns out to be equivalent to the search of binary matrices $A$ fulfilling that $AJ = dJ$ and $I + A + A^2 + A^3 + A^4 = J + P$, where $J$ denotes the all-one matrix and $P$ is a permutation matrix. Since the eigenvalues of $P$ are roots of unity, the factorization in $\mathbb{Q}[x]$ of the characteristic polynomial of $A$ involves the polynomials $F_n(x) = \Phi_n(1 + x + x^2 + x^3 + x^4)$, where $\Phi_n(x)$ denotes the $n$th cyclotomic polynomial. More precisely, if $F_n(x)$ is irreducible in $\mathbb{Q}[x]$ then it is a factor of $\det(xI - A)$ and its multiplicity only depends on the cycle structure of $P$. We conjecture that $F_n(x)$ is always irreducible in $\mathbb{Q}[x]$, unless $n = 1, 3, 6$. Under this assumption, we show how to derive the nonexistence of almost Moore digraphs of diameter four. Right now, by using tools from algebraic number theory, we have been able to prove that $F_n(x)$ for $n \neq 1, 3, 6$ is either irreducible or factorizes into two irreducible factors of degree $2\varphi(n)$.

Key words: Almost Moore digraph; Cyclotomic polynomial.

1 Introduction

It is well known that interconnection networks can be modeled by graphs whose vertices represent the processing elements and whose edges represent their links. The graphs thus obtained can be undirected or directed depending on whether the communication between nodes is two-way or only one-way. In this context the following problem arises quite naturally:

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• **Degree/diameter problem:** given two natural numbers \( d \) and \( k \), find the largest possible number of vertices \( N(d,k) \) in a [directed] graph with maximum [out-] degree \( d \) and diameter \( k \) (for a survey of it see [15]).

In the directed case, it has been proved that

\[
N(d,k) < 1 + d + \cdots + d^k = M(d,k),
\]

unless \( d = 1 \) or \( k = 1 \) (see [17,4]). Then, the question of finding for which values of \( d > 1 \) and \( k > 1 \) we have \( N(d,k) = M(d,k) - 1 \), where \( M(d,k) \) is known as the Moore bound, becomes an interesting problem. In this case, any extremal digraph turns out to be \( d \)-regular (see [12]). From now on, regular digraphs of degree \( d > 1 \), diameter \( k > 1 \) and order \( N = d + \cdots + d^k \) will be called almost Moore \((d,k)\)-digraphs (or \((d,k)\)-digraphs for short).

So far, the problem of the existence of almost Moore \((d,k)\)-digraphs has only been solved when \( d = 2, 3 \) or \( k = 2, 3 \). Thus, fixing the degree, Miller and Fris [13] proved that the \((2,k)\)-digraphs do not exist for values of \( k > 2 \) and, subsequently, Baskoro et al. [3] established the nonexistence of \((3,k)\)-digraphs unless \( k = 2 \). On the other hand, Fiol et al. [6] showed that the \((d,2)\)-digraphs do exist for any degree. The digraph constructed is the line digraph \( LK_{d+1} \) of the complete digraph \( K_{d+1} \). Concerning the enumeration of \((d,2)\)-digraphs it is known that there are exactly three non isomorphic \((2,2)\)-digraphs (see [14]). The classification of \((d,2)\)-digraphs was completed in [8] by proving that \( LK_{d+1} \) is the unique solution, if \( d \geq 3 \). Moreover, the authors proved the nonexistence of \((d,3)\)-digraphs in [5].

In this paper we study the existence of almost Moore digraphs of diameter \( k = 4 \). We prove their nonexistence, assuming the irreducibility in \( \mathbb{Q}[x] \) of the polynomials \( F_n(x) = \Phi_n(1 + x + x^2 + x^3 + x^4) \), whenever \( n \neq 1, 3, 6 \). Such polynomials appear in the factorization of the characteristic polynomial of a \((d,4)\)-digraph. Concerning the irreducibility of \( F_n(x) \), using concepts and techniques from algebraic number theory we have seen that \( F_n(x) \), when \( n \neq 1, 3, 6 \), is either irreducible or factorizes into two irreducible polynomials of degree \( 2\varphi(n) \). Nevertheless, our experimental results point out that \( F_n(x) \) is irreducible.

## 2 On \((d,k)\)-digraphs

Every \((d,k)\)-digraph \( G \) has the property that for each vertex \( v \in V(G) \) there exists only one vertex, denoted by \( r(v) \) and called the *repeat* of \( v \), such that there are exactly two \( v \rightarrow r(v) \) walks of length at most \( k \). If \( r(v) = v \), which means that \( v \) is contained in one \( k \)-cycle, \( v \) is called a *selfrepeat* of \( G \). The map \( r \), which assigns to each vertex \( v \in V(G) \) its repeat \( r(v) \), is an automorphism of \( G \) (see [1]). Seeing it as a permutation, \( r \) has a *cycle structure* which corresponds to its unique decomposition in disjoint cycles. Such cycles will
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be called permutation cycles of $G$. The number of permutation cycles of $G$ of each length $n \leq N$ will be denoted by $m_n$ and the vector $(m_1, \ldots, m_N)$ will be referred to as the permutation cycle structure of $G$.

Using the basic properties of a $(d,k)$-digraph $G$, it can be seen that its adjacency matrix $A$ fulfills the equation

$$I + A + \cdots + A^k = J + P,$$

where $J$ denotes the all-one matrix and $P = (p_{ij})$ is the $(0, 1)$-matrix associated with the permutation $r$ of $V(G) = \{1, \ldots, N\}$; that is to say, $p_{ij} = 1$ iff $r(i) = j$.

From Equation (1), the spectrum of $A$ and $J + P$ are closely related. It is known that the characteristic polynomial of $J + P$ is

$$\det(xI - (J + P)) = (x - (N + 1))(x - 1)^{m_1 - 1} \prod_{n=2}^{N} (x^n - 1)^{m_n}$$

(see [2]). Since $x^n - 1 = \prod_{i|n} \Phi_i(x)$, where $\Phi_i(x)$ denotes the $i$-th cyclotomic polynomial, the factorization of $\det(xI - (J + P))$ in $\mathbb{Q}[x]$ is

$$\det(xI - (J + P)) = (x - (N + 1))(x - 1)^{m(1) - 1} \prod_{n=2}^{N} \Phi_n(x)^{m(n)},$$

where $m(n) = \sum_{n|i} m_i$ represents the total number of permutation cycles of order multiple of $n$. In [7], the problem of the factorization in $\mathbb{Q}[x]$ of the characteristic polynomial of $G$, $\phi(G,x) = \det(xI - A)$, was connected with the study of the irreducibility in $\mathbb{Q}[x]$ of the polynomials

$$F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k).$$

The idea is that, when such polynomials are irreducible, then they are ‘big pieces’ of the characteristic polynomial of $G$.

**Proposition 1.** Let $(m_1, \ldots, m_N)$ be the permutation cycle structure of a $(d,k)$-digraph $G$ and $2 \leq n \leq N$. If $F_{n,k}(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$, then it is a factor of $\phi(G, x)$ and its multiplicity is $m(n)/k$.

Moreover, a conjecture about the irreducibility of $F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k)$, $n > 2$, in $\mathbb{Q}[x]$ was formulated in [7]. More precisely,

**Conjecture.** Let $n > 2$ and $k > 1$ be integers. One has that

(i) If $k$ is even, then $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $n \mid (k + 2)$,

(ii) If $k$ is odd, then $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $n$ is even and $n \mid 2(k + 2)$.
We will refer to this conjecture as the cyclotomic conjecture. The case $k = 2$ was proved by H.W. Lenstra Jr. and B. Poonen [11] and, recently, the authors proved the case $k = 3$ in [5]. Besides, the irreducibility of $F_{2,k}(x)$, for any $k$, was proved in [7].

We use the simplest spectral invariant, the trace of a matrix, in order to show that the equation $I + A + A^2 + A^3 + A^4 = J + P$ would have no $(0,1)$-matrix solutions such that $AJ = dJ$. More precisely, we derive a contradiction on some algebraic multiplicities of the eigenvalues of $A$ (see Section 4). We remark that instead of directly working with the eigenvalues of $A$, as it is usually done in spectral graph theory, we collect them into irreducible factors of the characteristic polynomial of $A$. Such a polynomial approach has also been used in the literature (see, for instance, [9,10]).

3 The cyclotomic conjecture for $k = 4$

This section is devoted to study this conjecture in the case $k = 4$. We show that the polynomial $F_{n,4}(x)$, when $n \neq 1,3,6$, is either irreducible in $\mathbb{Q}[x]$ or has only two factors of the same degree. From now on, we write $F_n(x)$ instead of $F_{n,4}(x)$.

As a first step, we prove that the condition of being $F_n(x)$ reducible in $\mathbb{Q}[x]$ either with more than two factors or with two factors of different degree implies a divisibility relation by a cyclotomic polynomial.

**Proposition 2.** Let $n \neq 1,3,6$ and $F_n(x) = \Phi_n(1+x+x^2+x^3+x^4)$. If $F_n(x)$ splits in $\mathbb{Q}[x]$ either with more than two factors or with two factors of different degree, then there exists a polynomial

$$p(z,y) = (1-z)^3 + (1-z)^2yz + (1-z)zy^2 + z^2 + (1-z)^3z^3y^4$$
$$- (6 - 8z - 3z^2)z^2y^5 + (1-z)(3 - 4z)z^2y^6 + 2(1 - 2z)z^2y^7$$
$$+ (1-z)^3z^4y^8 + 2(2 - z)z^4y^9 + (1-z)(4 - 3z)z^3y^{10}$$
$$+ (3 - 8z + 6z^2)z^3y^{11} + (1-z)^3z^3y^{12} - z^6y^{13}$$
$$+ (1-z)z^5y^{14} - (1-z^2)z^4y^{15} + (1-z)^3z^4y^{16}$$

such that

if $(n,2) = 1$ then $\Phi_{2n}(x)$ divides $p(x^2,x^4)$, $1 \leq \ell < 2n$,

if $(n,2) = 2$ then $\Phi_n(x)$ divides $p(x,x^\ell)$, $1 \leq \ell < n$.

**Proof.** For the cases $n = 2,4,5$ we can check that $F_n(x)$ is irreducible and for $n = 1,3,6$ it is reducible. Then, let us suppose that $n > 6$ and $F_n(x)$ is reducible in $\mathbb{Q}[x]$. Let us consider a root $\varepsilon$ of $F_n(x)$. Then $p_1(\zeta_n,\varepsilon) = 0$, where

$$p_1(z,w) = 1 - z + w + w^2 + w^3 + w^4,$$

(2)
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whith \( \zeta_n \) a primitive \( n \)-th root of unity. Using properties about the degrees of the algebraic extensions

\[ \mathbb{Q} \subseteq \mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(\varepsilon), \]

we derive that \( F_n(x) \) has an irreducible factor in \( \mathbb{Q}[x] \) of degree \( \varphi(n) \) or \( 2\varphi(n) \), where \( \varphi(n) \) stands for Euler’s function.

We can assume that \( \varepsilon \) is a root of such a factor. In particular, \( \varepsilon \) is an algebraic integer and \([\mathbb{Q}(\varepsilon) : \mathbb{Q}(\zeta_n)]\) is either 1 or 2. According to the hypothesis, this degree must be 1. Hence \( \varepsilon \in \mathbb{Z}[\zeta_n] \).

We seek for a polynomial relation between \( \alpha \) and \( \zeta_n \). In order to find such an expression we use the following identities:

\[ 1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^4 = \frac{\varepsilon^5 - 1}{\varepsilon - 1} = \zeta_n, \quad (3) \]

\[ \overline{\varepsilon} = \alpha \varepsilon. \quad (4) \]

From them, and taking into account that \( \overline{\zeta_n} = 1/\zeta_n \), it can be seen that

\[ \zeta_n = \frac{\alpha \varepsilon - 1}{\alpha^5 \varepsilon^5 - 1}. \quad (5) \]

By substituting \( \zeta_n \) in the identity (3) for the previous expression, it turns out that

\[ \varepsilon^4 = \frac{\zeta_n + \alpha}{\alpha^5 \zeta_n + 1}. \]

Hence \( \alpha^5 \zeta_n + 1 \neq 0 \). Otherwise \( \alpha = -\zeta_n \) and so \( \zeta_n^6 = 1 \), which corresponds to the cases \( n = 1, 2, 3, 6 \).

On the other hand, identity (5) can be written as \( p_2(\zeta_n, \alpha, \varepsilon) = 0 \) where

\[ p_2(z, y, w) = 1 - z - zyw - zy^2w^2 - zy^3w^3 - zy^4w^4. \quad (6) \]

Therefore, the relation between \( \zeta_n \) and \( \alpha \) we are looking for is \( p(\zeta_n, \alpha) = 0 \), where

\[ p(z, y) = \frac{1}{z - 1} \text{Res}(p_1(z, w), p_2(z, y, w), w). \quad (7) \]

Since \( \zeta_n - 1 \) is a prime or a unit in \( \mathbb{Z}[\zeta_n] \), by using a similar argument as in [5] we can deduced that \( \alpha \) is a unit. On the other hand, all conjugates of \( \alpha = \overline{\varepsilon}/\varepsilon \) have absolute value 1 and, from Lemma 1.6 of [19], we can derive that \( \alpha \) is a root of unity. Besides \( \alpha \neq 1 \) since otherwise \( \varepsilon \in \mathbb{R} \), which is a contradiction with the fact that Equation (2) has no real solutions. The order of \( \alpha \) as a root of unity can be \( 2n \) if \( (n, 2) = 1 \) or \( n \) if \( (n, 2) = 2 \).

Therefore we can take, in expression (7), \( \alpha = \zeta_{kn}^\ell (1 \leq \ell < kn) \) and \( \zeta_n = \zeta_{kn}^k \), where \( k \in \{ 1, 2 \} \) according to the previous cases. So, replacing \( \zeta_{kn} \) by \( x \) in (7) we obtain the polynomial equation \( p(x^k, x^\ell) = 0 \), which has a zero in \( x = \zeta_{kn} \). Consequently, \( \Phi_{kn}(x) \) must divide \( p(x^k, x^\ell) \).
Now we show that if $F_n(x)$, $n \neq 1, 3, 6$, is a reducible polynomial in $\mathbb{Q}[x]$ then it is a product of two irreducible factors of degree $2\varphi(n)$. Taking into account previous results, it is enough to see that $\Phi_{kn}(x) \mid p(x^k, x^\ell)$ for $k = 1, 2$, where

$$p(x^k, x^\ell) = -1 + 3x^k - 3x^{2k} + x^{3k} - x^{k+\ell} + 2x^{2k+\ell} - x^{3k+\ell} - x^{k+2\ell}$$

$$+ x^{2k+2\ell} - x^{k+3\ell} - x^{k+4\ell} + 3x^{2k+4\ell} - 3x^{3k+4\ell} + x^{4k+4\ell}$$

$$+ 6x^{2k+5\ell} - 8x^{3k+5\ell} + 3x^{4k+5\ell} - 3x^{2k+6\ell} + 7x^{3k+6\ell} - 4x^{4k+6\ell}$$

$$- 2x^{2k+7\ell} + 4x^{3k+7\ell} - x^{2k+8\ell} + 3x^{3k+8\ell} - 3x^{4k+8\ell} + x^{5k+8\ell}$$

$$- 4x^{4k+9\ell} + 2x^{5k+9\ell} + 4x^{3k+10\ell} - 7x^{4k+10\ell} + 3x^{5k+10\ell} - 3x^{3k+11\ell}$$

$$+ 8x^{4k+11\ell} - 6x^{5k+11\ell} - x^{3k+12\ell} + 3x^{4k+12\ell} - 3x^{5k+12\ell} + x^{6k+12\ell}$$

$$+ x^{6k+13\ell} - x^{5k+14\ell} + x^{6k+14\ell} + x^{4k+15\ell} - 2x^{5k+15\ell} + x^{6k+15\ell}$$

$$- x^{4k+16\ell} + 3x^{5k+16\ell} - 3x^{6k+16\ell} + x^{7k+16\ell}.$$

**Proposition 3.** If $F_n(x)$, $n \neq 1, 3, 6$, is a reducible polynomial in $\mathbb{Q}[x]$ then it is a product of two irreducible factors of degree $2\varphi(n)$.

**Proof.** From Proposition 2, it is enough to show that for $n \neq 1, 3, 6$, the polynomial $\Phi_{kn}(x)$ does not divide the polynomial $G(x) = p(x^k, x^\ell)$, for $k \in \{1, 2\}$, $1 \leq \ell < kn$ and for all integer $n$.

Let us suppose $\Phi_{kn}(x) \mid p(x^k, x^\ell)$ for some $k \in \{1, 2\}$ and for some $1 \leq \ell < kn$. We will prove that $\varphi(n)$ is bounded. Indeed, the polynomial $p(y, z)$ has degree 7 in $z$ and degree 16 in $y$. We can consider the polynomial

$$p_1(z, y) = k \frac{\partial p}{\partial z}(z, y) z + \ell \frac{\partial p}{\partial y}(z, y) y \in \mathbb{Z}[z, y].$$

Obviously $p_1(x^k, x^\ell) = x G'(x)$, where $G'(x)$ denotes the derivative of $G(x)$ with respect to $x$. Let $p$ be a prime integer dividing $nk$ with $nk = p^m m$ and $(m, p) = 1$. From part (i) of Lemma 3 in [5], we know that

$$\Phi_m(x)^{\varphi(p^m) - 1} \mid \gcd (G(x), x G'(x)) \pmod {p \mathbb{Z}[x]}$$

and therefore

$$\Phi_m(x)^{\varphi(p^m) - 1} \mid P_{\ell,k}(x^k) \pmod {p \mathbb{Z}[x]},$$

where $P_{\ell,k}(z)$ is the following resultant

$$P_{\ell,k}(z) = \text{Res} (p(z, y), p_1(z, y), y).$$

More precisely,

$$P_{\ell,k}(z) = z^{64} \Phi_1(z)^{30} \Phi_2(z)^{12} \Phi_3(z)^6 \Phi_6(z)^6 Q_{\ell,k}(z),$$

where $Q_{\ell,k}(z)$ is a polynomial of degree 30.
The polynomial $Q_{\ell,k}(z) \equiv 0 \pmod{p\mathbb{Z}[x]}$ if and only if $p = 5$ and $k \equiv \ell \pmod{5}$ or $p = 2, 3$ and $k \equiv \ell \equiv 0 \pmod{3}$. Otherwise, $\varphi(n) \leq 60 \cdot 8 + 1 = 481$.

Assume that $p = 3$ and $k \equiv \ell \equiv 0 \pmod{3}$. In this case

$\Phi_{n, k/3}(x) \mid p(x^{k/3}, x^{\ell/3})$.

If $e > 1$, then $\varphi(n) \leq 481$ and for $e = 1$ we have that $3 \not| (n k/3)$ and we can discard $p = 3$.

Assume that $n = 5^r$ with $r \geq 1$. We will prove that $F_n(x)$ is irreducible. Indeed,

$$F_n(x) = x^{\varphi(n)} + \sum_{i=0}^{\varphi(n)-1} a_i \equiv ((x - 1)^4 - 1)^{\varphi(n)} \pmod{5\mathbb{Z}[x]}$$

$$\equiv (x(x + 1)(x + 2)(x + 3))^{\varphi(n)} \pmod{5\mathbb{Z}[x]},$$

which implies that $a_i \equiv 0 \pmod{5}$ for all $i$. Since $a_0 = 5 \not\equiv 0 \pmod{5^2}$, by Eisenstein’s criterion it follows that $F_n(x)$ is irreducible in $\mathbb{Q}[x]$.

Now, we have only to bound $\varphi(n)$ when $n = 2^s 5^r$ with $s \geq 1$, $k = 2$ and $\ell$ is even. Therefore, $\Phi_{n}(x)$ divides $p(x, y^{\ell/2})$. Since $Q_{\ell/2,1}(z)/z^4$ is a polynomial of degree 22, it follows that $\varphi(n) \leq 62 \cdot 4 + 1 = 249$.

For these particular values of $n$, it has been computationally checked that $F_n(x)$ is irreducible in $\mathbb{Q}[x]$ unless $n \neq 1, 3, 6$.

4 Nonexistence of almost Moore digraphs of diameter four

As we have already mentioned, the irreducibility of the polynomials $F_n(x)$ plays a key role in the factorization of the characteristic polynomial of a $(d, 4)$-digraph $G$. Their corresponding multiplicities depend on the permutation cycle structure of $G$ and they are uniquely determined, apart from the (three) cases where $F_n(x)$ is reducible (according to Conjecture 2 such cases are $n = 1, 3, 6$). Then, by computing some spectral invariants of $G$, as they are the traces of the first three powers of its adjacency matrix $A$, we would be able to conclude that some of the unknown multiplicities must be negative, which would provide us a proof of the nonexistence of $G$.

**Theorem 1.** Assuming that the cyclotomic conjecture is true for $k = 4$, there is no almost Moore digraph of diameter four.

**Proof.** Let $G$ be a $(d, 4)$-digraph of degree $d > 3$ and let $(m_1, \ldots, m_N)$ be its permutation cycle structure, where $N = d + d^2 + d^3 + d^4$.

First, we obtain the factorization of the characteristic polynomial $\phi(G, x)$ of $G$. Since for any integer $n > 1$ and $n \neq 3, 6$ the polynomial $F_n(x)$ is irreducible in $\mathbb{Q}[x]$ (see Theorem 3), applying Proposition 1 we have that
\[
\prod_{2 \leq n \leq N; n \neq 3, 6} (F_n(x))^{\frac{m(n)}{4}} \text{ is a factor of } \phi(G, x).
\]

The remaining factors of \(\phi(G, x)\) are derived as follows:

- Since \(G\) is \(d\)-regular and strongly connected, \(\phi(G, x)\) has the linear factor \(x - d\) with multiplicity 1;
- Taking into account that \(x - 1\) is a factor of \(\det(xI - (J + P))\) with multiplicity \(m(1) - 1\) and since
  \[F_1(x) = (x + 1)(x^2 + 1)x,\]
  we have that \(x + 1, x^2 + 1\) and \(x\) are factors of \(\phi(G, x)\) with multiplicities \(a_1, a_2\) and \(a_3\), respectively, where \(a_1 + 2a_2 + a_3 = m(1) - 1\);
- Since \(\Phi_3(x) = x^2 + x + 1\) is a factor of \(\det(xI - (J + P))\) with multiplicity \(m(3)\) and taking into account the factorization of \(\Phi_3(x)\) in \(\mathbb{Q}[x]\),
  \[F_3(x) = (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3),\]
  we have that \(\Phi_6(x) = x^2 - x + 1\) and \(\Phi_3(x)/\Phi_6(x)\) are factors of \(\phi(G, x)\) with multiplicities \(b_1\) and \(b_2\), respectively, where \(2b_1 + 6b_2 = 2m(6)\); that is, \(b_1 = m(3) - 3b_2\). Analogously, since the factorization of \(F_6(x)\) in \(\mathbb{Q}[x]\) is
  \[F_6(x) = (x^2 + x + 1)(x^6 + x^5 + x^4 + 2x^3 + x^2 + 1),\]
  we have that \(\Phi_3(x)\) and \(F_6(x)/\Phi_3(x)\) are factors of \(\phi(G, x)\) with multiplicities \(c_1\) and \(c_2\), respectively, where \(c_1 = m(6) - 3c_2\).

As a result,
\[
\phi(G, x) = (x - d)(x + 1)^{a_1}(x^2 + 1)^{a_2}x^{a_3}\Phi_6(x)^{b_1}(F_6(x)/\Phi_3(x))^{b_2} \prod_{2 \leq n \leq N; n \neq 3, 6} (F_n(x))^{\frac{m(n)}{4}}. \quad (10)
\]

Next, we compute the graph spectral invariants \(\text{Tr } A^\ell \ (\ell = 1, 2, 3)\) in terms of the sum of the \(\ell\)-th powers of the roots of each factor of \(\phi(G, x)\).

Given a monic polynomial of degree \(n \geq 1\), \(a(x) = x^n + \sum_{i=1}^{n} a_{n-i}x^{n-i}\) and given an integer \(\ell \geq 1\), we define \(S_\ell(a(x))\) to be the sum of the \(\ell\)-th powers of all the roots of \(a(x)\). Using Newton’s formulas [18], which express \(S_\ell(a(x))\) in terms of the coefficients of \(a(x)\), we have
\[
S_1(a(x)) = -a_{n-1}
S_2(a(x)) = a_{n-1}^2 - 2a_{n-2}
S_3(a(x)) = -a_{n-1}^3 + 3a_{n-1}a_{n-2} - 3a_{n-3}.
\]

In particular, taking into account that
Besides, it can be easily checked that and since where 

Thus, of all factors of \( \phi(G,x) \).

Now, for each \( \ell = 1, 2, 3 \) we can express the trace of the \( \ell \)th power of the adjacency matrix \( A \) of \( G \) in terms of the sums \( S_\ell \) of all factors of \( \phi(G,x) \). Thus,

\[
\text{Tr} \, A = d - a_1 + b_1 - 3b_2 - c_1 - c_2 - \frac{1}{4} T,
\]

\[
\text{Tr} \, A^2 = d^2 + a_1 - 2a_2 - b_1 - b_2 - c_1 - c_2 - \frac{1}{4} T,
\]

\[
\text{Tr} \, A^3 = d^3 - a_1 - 2b_1 + 2c_1 - 4c_2 - \frac{1}{4} T,
\]

where \( T = \sum_{2 \leq n \leq N} m(n) \varphi(n) \). From the identity \( \sum_{n=1}^{N} m(n) \varphi(n) = N \) (see [7]),

\[
T = N - m(1) - 2m(3) - 2m(6).
\]

So, taking into account that \( b_1 = m(3) - 3b_2 \) and \( c_1 = m(6) - 3c_2 \),

\[
\text{Tr} \, A = d - \frac{1}{4} N + \frac{1}{4} m(1) + \frac{3}{2} m(3) - \frac{1}{2} m(6) - a_1 - 6b_2 + 2c_2
\]

\[
\text{Tr} \, A^2 = d^2 - \frac{1}{4} N + \frac{1}{4} m(1) - \frac{1}{2} m(3) - \frac{1}{2} m(6) + a_1 - 2a_2 + 2b_2 + 2c_2
\]

\[
\text{Tr} \, A^3 = d^3 - \frac{1}{4} N + \frac{1}{4} m(1) - \frac{3}{2} m(3) + \frac{5}{2} m(6) - a_1 + 6b_2 - 10c_2.
\]

Since \( G \) has no cycles of length \( \leq 3 \), we know that \( \text{Tr} \, A^\ell = 0 \) (\( \ell = 1, 2, 3 \)). As a consequence,

\[
4a_1 + 24b_2 - 8c_2 = 4d - N + m(1) + 6m(3) - 2m(6)
\]

\[
-4a_1 + 8a_2 - 8b_2 - 8c_2 = 4d^2 - N + m(1) - 2m(3) - 2m(6)
\]

\[
4a_1 - 24b_2 + 40c_2 = 4d^3 - N + m(1) - 6m(3) + 10m(6).
\]

Applying Gauss reduction method to the previous linear system, it follows that
\begin{align*}
8a_2 + 16b_2 - 16c_2 &= 4d^2 + 4d - 2N + 2m(1) + 4m(3) - 4m(6) \quad (11) \\
-48b_2 + 48c_2 &= 4d^3 - 4d - 12m(3) + 12m(6). \quad (12)
\end{align*}

Taking into account that \( N = d^4 + d^3 + d^2 + d \), from (11) and (12) we derive that
\[24a_2 = 4d^3 + 12d^2 + 8d + 6m(1) - 6N.\]

Notice that \( m(1) = \sum_{n=1}^{N} m_n \) takes its maximum value when all permutation cycles are short as possible. Moreover, the number of selfrepeats \( m_1 \) of a \((d, k)\)-digraph is either 0 or \( k \), if \( k \geq 3 \) (see [1]). So, \( m(1) \leq 4 + \frac{N-1}{2} \) and, consequently,
\[24a_2 \leq 4d^3 + 12d^2 + 8d + 12 - 3N = -3d^4 + d^3 + 9d^2 + 5d + 12\]

Hence, if \( d > 3 \) then \( a_2 < 0 \), which is impossible since \( a_2 \) is a nonnegative integer.

References


On the nonexistence of almost Moore digraphs of diameter four


