Minimal strong digraphs

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Abstract. We introduce adequate concepts of expansion of a digraph and we characterize the class of minimal strong digraphs whose expansion preserves the property of minimality. We prove that every minimal strong digraph of order \( n \geq 2 \) is the expansion of a minimal strong digraph of order \( n - 1 \) and we give sequentially generative procedures for the constructive characterization of the classes of minimal strong digraphs and strong digraphs. Finally we describe algorithms to compute unlabeled minimal strong digraphs and isospectral classes of minimal strong digraphs.

Key words: (minimal) strong digraphs, isospectral strong digraphs.

1 Introduction

In this article, we focus on the study of strongly connected digraphs containing the least possible number of arcs (minimal strong digraphs), that is, strongly connected digraphs which cease to be so if any one of its arcs is suppressed.

We are previously interested in the following problem [6]: given a real polynomial \( p(x) \) with degree \( n \), find necessary and sufficient conditions for the existence of a nonnegative matrix \( A \) of order \( n \) with characteristic polynomial \( p(x) \). The irreducible realizations of \( p(x) \) are identified with strongly connected digraphs associated to \( A \) [2]. The class of strong digraphs can easily be reduced to the class of minimal strong digraphs, so we are interested in this one.

Many classes of connected graphs and digraphs have constructive characterizations. In particular, for (minimal) 2-connected graphs and (minimal) strong digraphs different procedures have been described to construct larger (di)graphs from smaller (di)graphs [4,5,1]. The common basic idea of these procedures consists of adding paths between qualified vertices.

Bhogadi [1] gives a characterization of Cunningham’s decomposition trees for minimal strong digraphs under X-joint composition [3] and uses this characterization to catalogue all the Cunningham prime minimal strong digraphs through 11 vertices and to implement an algorithm, for the generation of all minimal strong digraphs

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through 12 vertices. This approach was abandoned due to the difficulty of recognizing prime minimal strong digraphs.

All of these procedures have been defined so that the conditions under which minimality is preserved are not characterized. This is not a difficulty when proving the possibility of obtaining any minimal strong digraph from another smaller one (Hedetneimi [5] gives a proof), but it is a difficulty when constructing efficient and sequential procedures and algorithms.

2 Results

We introduce two suitable concepts of expansion of a digraph for a sequential construction of minimal strong digraphs and characterize the class of minimal strong digraphs whose expansion preserves the property of minimality. We show how every minimal strong digraph of order $n \geq 2$ is the expansion of a minimal strong digraph of order $n - 1$ and describe procedures to do this.

We propose a sequentially generative procedure for the constructive characterization of the class of minimal strong digraphs and we give a simple procedure to build the set of strong digraphs of order $n \geq 3$ starting from the set of minimal strong digraphs of order $n - 1$.

We implement an algorithm to compute unlabeled minimal strong digraphs following the construction of the previous sections. Another algorithm allows the digraphs and the characteristic polynomials of the isospectral classes of the minimal strong digraphs to be obtained.

References


