2021 GAiA

Titles and abstracts for the short presentations (in the order of the schedule)

January 26, 2021

Manuel Lainz. *The geometry of Herglotz's principle of least action*

Abstract:

A trajectory c will be followed by the system if and only if c is a critical point of the action among all trajectories with the same endpoints as c." If by the *action* we mean the integral of a Lagrangian function L along c and its velocity, this is *Hamilton's least action principle* which can be used to describe most physical theories of mechanics in which some energy is preserved, and has many applications in other fields. Its solutions have a nice geometric characterization: they are integral curves of a Hamiltonian vector field on a symplectic manifold.

Even more systems can be described if we generalize the principle of least action: the so-called *Herglotz's principle*. Here, the Lagrangian not only depends on the positions and velocities, but also on the action itself. Hence, the action is no longer the integral of the Lagrangian, but it is the solution of a non-autonomous ODE. This principle allows us to model new problems, such as some dissipative systems in mechanics (where energy is lost), thermodynamics, and some modified optimal control systems. We also have a similar geometric characterization, but this time we have to use contact manifolds, which are an odd dimensional analogs of symplectic ones. However, this change on the geometry will imply some important differences on the dynamical behavior of the solutions.

Jacob Goodman. *Decentralized Motion Control of Multiagent systems on Riemannian manifolds*

Abstract:

In this talk, I introduce a variational approach for decentralized collision avoidance of multiple agents evolving on a complete Riemannian manifold. I begin by deriving necessary conditions for extremal, which consists of finding non-intersecting trajectories for a given number of agents sharing only the information of relative positions with respect to their nearest neighbors—among a set of admissible curves—such that these trajectories are minimizers of an energy functional. The energy functional depends on covariant acceleration and an artificial potential used to prevent collision among the agents. I then prove the global existence of sufficiently regular extrema for the energy functional. Finally, I apply the results to the case of agents on a compact and connected Lie group. Simulations are shown to demonstrate the applicability of the results. This is a joint work with L. Colombo.

Damir Ferizović. *Potential Theory with Multivariate Kernels I*

Abstract: In the first part of this talk I will give a small introduction to K-energies of point sets $\omega_N = \{x_1, \ldots, x_N\} \subset \Omega$, i.e. sums of the form

$$\sum_{i \neq j}^{n} K(x_i, x_j);$$

where K is a nice enough real valued kernel on a compact set Ω, like the Coulomb and Riesz s-energies on the sphere. This part will be (mostly) based on [1], and we will see how energies are related to the surface area via asymptotics of certain energies. We will introduce important concepts like (conditionally) positive definiteness, and present some celebrated theorems.

The second part deals with the generalization to n-input kernels, where for the sake of simplicity we will only present the case $n = 3$. We will give some motivation on why this might be interesting and generalize concepts from the two-input case to our setting. We will prove some simple consequences and set the stage for the sister talk given by Ryan Matzke.

This talk is based on [2], a joint work with D. Bilyk, A. Glazyrin, R. Matzke, J. Park, and O. Vlasiuk.

[1] J. Brauchart and P. Grabner: *Distributing many points on spheres: Minimal energy and designs.* Journal of Complexity, Vol 31, (3); pp 293-326 (2015).

[2] D. Bilyk, D. Ferizović, A. Glazyrin, R. Matzke, J. Park, and O. Vlasiuk: *Potential Theory with Multivariate Kernels.* Work in progress (2021).

Ryan Matzke. *Potential Theory with Multivariate Kernels II*

Abstract: In the past century, a great deal of study has been devoted to energy optimization problems. Given a compact metric space Ω and a kernel $K : \Omega^2 \to \mathbb{R} \cup \{\infty\}$, these problems involve finding points sets $\omega_N = \{z_1, ..., z_N\}$ in Ω or Borel probability measures μ on Ω that minimize/maximize the discrete energy

$$E_K(\omega_N) = \frac{1}{N^2} \sum_{j,k=1}^{N} K(z_j, z_k)$$

or the continuous energy

$$I_K(\mu) = \int_\Omega \int_\Omega K(x, y) d\mu(x) d\mu(y),$$

respectively. Such optimization problems can have applications in signal processing, discrepancy theory, discretization of manifolds, discrete geometry, and models of physical phenomena, a classical example coming from the Thomson problem from 1904, which asks what configurations minimize the electrostatic potential energy of N electrons on the sphere.

While a wealth of theory has been developed for classical energies, which model pairwise interactions between particles, of the above forms, there as been little to no systematic study of multivariate energies, defined by interatctions of triples, quadruples, or even higher numbers of particles, i.e. energies of the type

$$E_K(\omega_N) = \frac{1}{N^n} \sum_{j_1,...,j_n=1}^{N} K(z_{j_1}, ..., z_{j_n})$$

and

$$I_K(\mu) = \int_\Omega \cdots \int_\Omega K(x_1, ..., x_n) d\mu(x_1)...d\mu(x_n),$$

where $n \geq 3$ and $K : \Omega^n \to \mathbb{R} \cup \{\infty\}$. While less common than energies involving a two-input kernel, multivariate energies have appeared in various settings, including geometric measure theory, material science, and the determination of the exact kissing number in dimension 4.

In this presentation, we discuss current progress in developing theory for multivariate energy optimization, such as semi-definite programming and the consequences of a generalization of positive definiteness. We will also discuss some applications, such as maximizing the expected area squared of a random triangle with vertices on the unit sphere.

This the second part of a joint presentation with Damir Ferizović (Graz University of Technology) and is based on our joint work with Dmitriy Bilyk (University of Minnesota), Alexey Glazyrin (University of Texas Rio Grande Valley), Josiah Park (Texas A&M University), and Oleksandr Vlasiuk (Florida State University).

Pablo Gómez. *Comparison Theory in Riemannian Geometry*

Abstract:

The results inside the set of statements in differential geometry known as "comparison theorems" show a common pattern: a riemannian manifold with particular characteristics is collated with a specific standard space of constant curvature and, from this comparison, we seek conclusions related to the structure of the first manifold.

In this talk, we deal with the exposition of several comparison theorems in riemannian geometry. In particular, the results whose geometrical meaning is intended to assimilate are the Rauch Theorems, related with the Jacobi fields over a manifold; the Bishop-Gromov Inequality, that studies the volume growth of riemannian balls; the Toponogov Triangle Theorem, that compares distances between points in a triangle; and the Cheeger-Gromoll Soul Theorem, that states that there is a particular submanifold inside an open manifold with nonnegative curvature.

María Martín. *Completeness and stability of spacetimes with Lorentz-Minkowski ends*

Abstract:

Starting from the sufficient conditions of completeness and stability of completeness of Lorentzian manifolds, we find a result in the completeness of a specific type of Lorentzian manifolds in which these conditions can be relaxed. These manifolds are defined as spacetimes with Lorentz-Minkowski ends. The result can be improved using the criteria of completeness of compact manifolds that have precompact holonomy.

Alexandre A. Simones. *The exponential map for second order differential equations*

Abstract:

One of the most useful constructions in Riemannian geometry is the exponential map, which allows to identify at least small neighbourhoods of a Riemannian manifold with a vector space. This construction has several applications as, for instance, the introduction of normal coordinates or the geometric proofs of minimization properties of geodesics. We generalize the exponential map to the case of arbitrary second order differential equations and prove that it is a diffeomorphism in a neighbourhood of the return vector. We discuss a particular application to the setting of nonholonomic mechanics, where an exponential map may be constructed to prove that the attainable set of points reached by following nonholonomic trajectories is locally diffeomorphic to the constraint distribution.

José Manuel Fernández Barroso. *Can one hear geometric properties related with the Jacobi operator on closed Riemannian manifolds?*

A geometrical property is said to be audible if it can be determined from the eigenvalues of the Laplace–Beltrami operator. In this sense, there are audible properties, for example the volume of the manifold or the total scalar curvature. On the hand, there are properties which have been proved to be inaudible, this is the case of the D'Atri property, which is a generalization of the locally symmetry in the sense that the local geodesic symmetries are volume preserving, up to sign. Equivalently, D'Atri spaces was characterize by the Ledger conditions. These conditions are related with the Riemannian curvature operator and more precisely, with the trace of the Jacobi operator and its covariant derivatives. Another inaudible symmetric–like property is the type $\mathcal{A}$ property, this is $(\nabla_X \mathrm{Ric})(X, X) = 0$, where Ric is the Ricci tensor given by the trace of the Jacobi operator.

In our work, we continue studying the audibility of geometric properties related with the Jacobi operator on closed Riemannian manifolds.

This is a joint work with Teresa Arias-Marco.